Manifold Structures of Unstable Periodic Orbits and the Appearance of a Periodic Window in the Lorenz system (Nonlinear Dynamics in Macro-economics)

Kobayashi, Miki U.; Saiki, Yoshitaka

数理解析研究所講究録
第 1713 巻 2010 年 35-43

Manifold Structures of Unstable Periodic Orbits and the Appearance of a Periodic Window in the Lorenz system

Miki U. Kobayashi, Yoshitaka Saiki
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.
小林幹, 斉木吉隆
京都大学数理解析研究所

abstract

Manifold structures of the Lorenz system with sets of non-classical parameter values are investigated in terms of unstable periodic orbits embedded in the attractor. It is found that the structures are determined approximately by many unstable periodic orbits embedded in the attractor. Furthermore, an angle between a stable manifold and an unstable manifold of an unstable periodic orbit, which is measured by using covariant Lyapunov vectors, characterizes a parameter at which a periodic window related to the unstable periodic orbit emerges. In particular, when an unstable periodic orbit at some parameter has low angle (high angle) between a stable manifold and an unstable manifold, the periodic window corresponding to the unstable periodic orbit exists near (away from) the parameter. Due to this fact, the window sequence in a parameter space is almost determined from the information at a parameter, even if a window is quite small.

1 Introduction

Chaotic dynamical systems are interesting research fields which are studied not only mathematics, physics, engineering and biology but also economics [1, 2, 3]. Chaotic dynamical systems are divided into two groups from a viewpoint of manifold structures, i.e. hyperbolic systems and non-hyperbolic systems [4, 5]. A dynamical system is said to be hyperbolic if the stable and unstable manifolds are everywhere transversal to each other; otherwise a system is non-hyperbolic.

It is expected that most of mathematical models of actual phenomena are non-hyperbolic. It is, however, difficult to know whether the models are hyperbolic or non-hyperbolic, and thus the research judging non-hyperbolicity is scarce. In mathematics, hyperbolic systems have been studied well and are currently getting understood well at least from the view point of geometrical aspects [6]. On the other hand, non-hyperbolic systems are less explored fields and one of the most interesting research fields in dynamical systems.

It is known that unstable periodic orbits are powerful tools to analyze hyperbolic chaotic dynamical systems. For example, an useful formula is proposed to approximate various static quantities of systems, e.g. invariant measures and Lyapunov dimensions [7, 8, 9, 10]. The formula can be applied to several non-hyperbolic chaotic systems [7, 11]. On the other hand, it is also found that without modifications this does not work for the Lorenz system with a set of non-classical parameter values [12]. Generally, how we should modify the formula is an open problem. Furthermore, it is known that if the number of periodic orbits changes when a system is perturbed, the system is non-hyperbolic. Periodic orbits are also useful tools to determine whether the system is hyperbolic or non-hyperbolic [13].

In chaotic dynamical systems, there are parameter regions where strange attractors vanish and stable periodic orbits emerge. The parameter region is named a periodic window and exists in various chaotic systems. A construction of a periodic window is often related to the saddle-node bifurcation. A stable periodic orbit in a periodic window collides with an unstable periodic orbit at an edge of the window and the two periodic orbits disappear. At a time, a region of the periodic window vanishes away. Therefore, we can take a natural idea that the edge of a periodic window is non-hyperbolic, in other words, non-hyperbolic regions are closely related to periodic windows. However, it is not known what kind of non-hyperbolicity occures at the points, that is, non-hyperbolicity with tangency structures or without ones. Our goal in this paper is to clear the problem.

In this paper, we analyze the relation between manifold structures of unstable periodic orbits and periodic windows in the Lorenz system with sets of non-classical parameter values. It is known that the Lorenz system with a set of the classical parameter values is (singular) hyperbolic [5, 14, 15], and by changing a parameter, the system becomes non-hyperbolic [12, 16, 17, 18]. We focus manifold structures of unstable periodic orbits and discuss the relation between the structures and periodic windows. Furthermore, we propose a method to identify positions of periodic windows from the manifold structures at some parameter value.

We use covariant Lyapunov vectors (CLVs) to investigate manifold structures in the Lorenz system, especially the angle between a stable manifold and an unstable manifold of each unstable periodic orbit. The CLVs span the Oseledec subspaces corresponding to each Lyapunov exponent [19, 20, 21].

2 Periodic windows as the origin of non-hyperbolicity

In this section, we discuss manifold structures in the Lorenz system:

$$\dot{x} = -\sigma x + \sigma y, \dot{y} = -xz + rx - y, \dot{z} = xy - bz. \tag{1}$$

In this paper, we fix $\sigma = 10, b = 8/3$ and treat r as a control parameter. In particular, we use parameter region in the neighborhood of $r = 60$, It is found that the Lorenz system is hyperbolic at $r = 28$ and becomes non-hyperbolic as r increases [17, 22]. Here we study the origin of non-hyperbolicity in the Lorenz system around $r = 60$. There are some studies conjecturing that the parameter region around $r = 60$ is non-hyperbolic [12, 16].

Figure 1 shows the bifurcation diagram of local maximal values of z in the time development for each control parameter r and in this region chaotic attractors exist. As is the case with general chaotic dynamical systems, there are quite many periodic windows in the Lorenz system. As the region of most periodic windows, however, are very small, it is difficult to detect them (see Fig. 1 (right)).

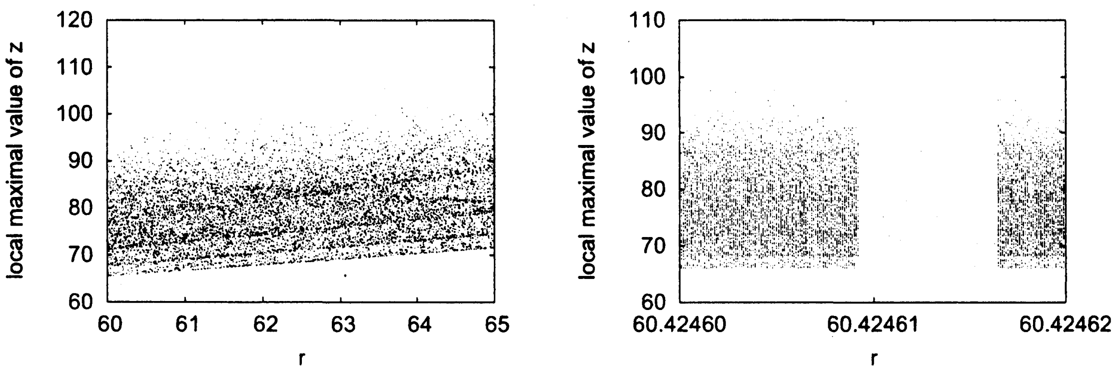

FIG. 1: Bifurcation diagram of the Lorenz system: (left) : $60 < r < 65$. (right) : $60.42460 < r < 60.42462$. There is a quite narrow $X^4Y^1X^3Y^3$ periodic window that we can find only after scale-up.

A periodic window in the Lorenz system emerge via an inverse period doubling bifurcation and finishes by a saddle-node bifurcation [22]. In this paper, we will continue to make use of the following symbolic description of periodic orbits: we will write an X everytime the orbit spirals round in $x > 0$ and a Y everytime it spirals round in $x < 0$ [22]. A window shown in Fig. 1 (right) is the $X^4YX^3Y^3$ window which finishes when saddle and node of $X^4YX^3Y^3$ periodic orbits collide. As is seen from Fig. 1 (right), most periodic windows are very small. The number of unstable periodic orbits changes at the end of a periodic window where saddle-node bifurcation occurs. This implies that the end of a periodic window is a non-hyperbolic parameter from mension above. We substantiate this idea on a rigorous basis studying the CLVs (we will write CLVs at time n as $\boldsymbol{v}_n$).

The knowledge of the CLVs allows estimating hyperbolicity or non-hyperbolicity by determining the angle between each pair (j, k) of expanding (j) and contracting (k) directions, $\phi_n^{j,k} = \cos^{-1}(|\boldsymbol{v}_n^j \cdot \boldsymbol{v}_n^k|) \times 180/\pi$ [19]. Remark that we will sometimes call an angle between a stable and an unstable manifolds as just an angle. Figure 2 shows the probability density function $\rho(\phi)$ of an angle between a tangent vector of a stable manifold $(\boldsymbol{v}_n^3)$ and a tangent vector of an unstable manifold $(\boldsymbol{v}_n^1)$. The distribution is bounded away from 0 (degree) for the case of $r = 28$, while arbitrarily small angles are found for the case of $r = 60$. From the results, we can conclude that the region $r = 28$ is hyperbolic, and $r = 60$ is non-hyperbolic. In Figure 3 (right), we show points on a chaotic orbit whose angles are less than 0.5 (degree). From this figure, it is expected that tangency structures of stable and unstable manifolds are somewhere around the points and embedded approximately in the edges of the strange attractor in this parameter.

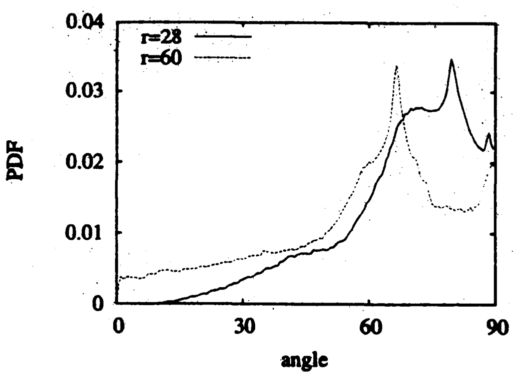

FIG. 2: The distribution of angles between stable and unstable manifolds at points on a chaotic attractor for r = 28(dashed line), r = 60(full line). The system at r = 28 is hyperbolic, while r = 60 is non-hyperbolic.

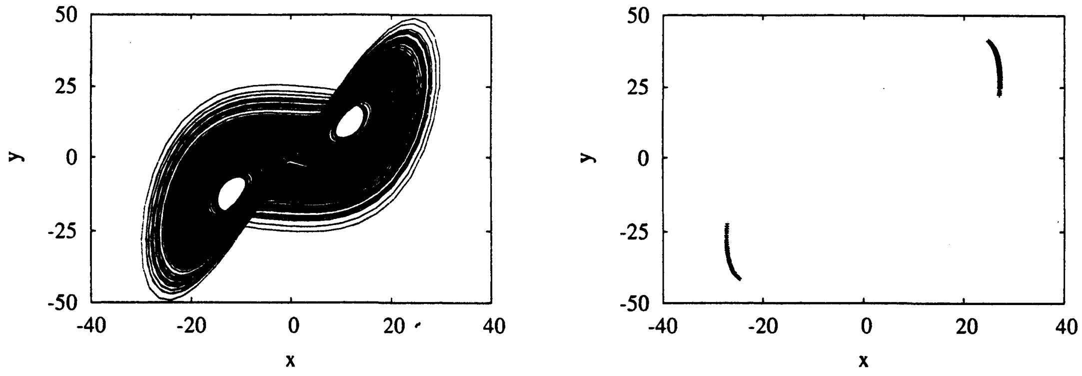

FIG. 3: (right) A chaotic orbit with $r = 60$ projected onto the $x - y$ plane. (left) Points on the orbit whose angles are less than 0.5(degree) projected onto the $x-y$ plane. The tangency structures are around the points.

Properties of chaos can be characterized in terms of unstable periodic orbits embedded in the attractor [7, 8, 9, 10]. We think that manifold structures can also be characterized in terms of unstable periodic orbits. We calculate angles between stable and unstable manifolds of unstable periodic orbits to characterize an unstable periodic orbit.

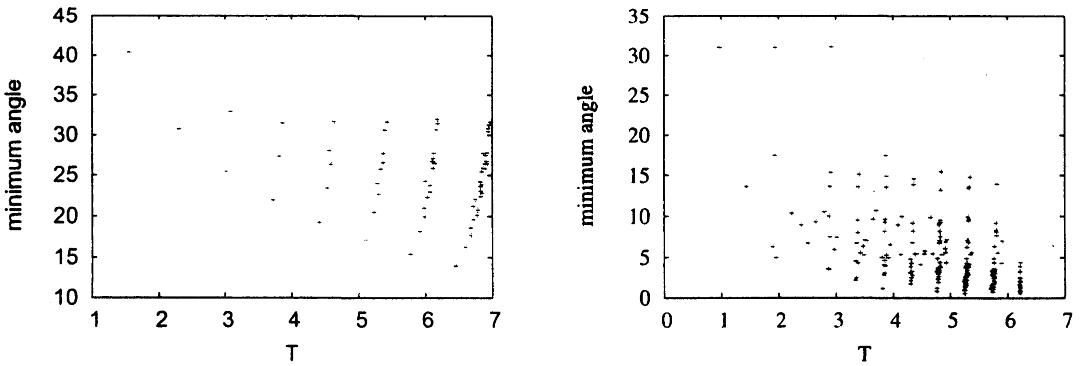

FIG. 4: Minimum angle between stable and unstable manifolds at points on each of numerically detected hundreds of unstable periodic orbits at (a) $r = 28$ and (b) $r = 60$. longitudinal axis : period of unstable periodic orbits. horizontal axis : minimum angle.

Figure 4 shows a minimum angle between stable and unstable manifolds at points on each of numerically detected hundreds of unstable periodic orbits at $r = 28$ and $r = 60$. It is found that unstable periodic orbits in the case of $r = 28$ tend to have large angles, whereas some unstable periodic orbits in $r = 60$ have very small angles. This result suggests that non-hyperbolicity is characterized by unstable periodic orbits which have small angles, that is, unstable periodic orbits which pass near tangency structures. We can study tangency structures of chaos by using these unstable periodic orbits.

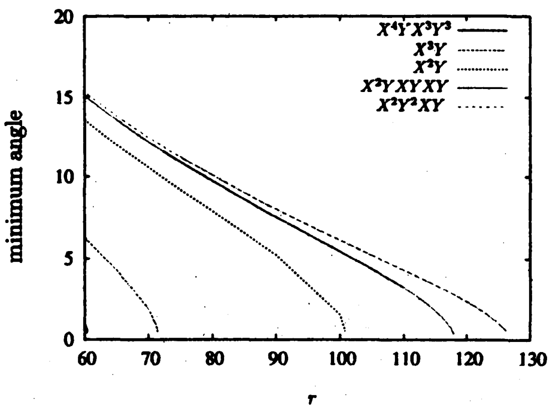

FIG. 5: Parameter dependency on angles between stable and unstable manifolds of several unstable periodic orbits, $X^4YX^3Y^3$, X^3Y, X^2Y, X^2YXYXY and X^2Y^2XY.

To see the changes of manifold structures of unstable periodic orbits, we measure the angles of five unstable periodic orbits which are detected at $r = 60$ as parameter changes(Fig.5). Fig.5 shows that the minimum angle for each unstable periodic orbit becomes the minimum value at the end of the correponding periodic window. This fact indicates that there is at least one unstable periodic orbit with small angle at the end of the periodic window. We should note that the minimum angle at the ending point of each periodic window is not exact zero but small positive value. In conclusion, the end of each periodic window is a non-hyperbolic system without any tangency structures.

Fig. 5 also shows that the minimum angle decreases monotonically and approaches to zero value as r increases, and a sequence of minimum angles of 5 unstable periodic orbits holds for any parameter r. This indicates that if there is an unstable periodic orbit which has a small angle at a parameter, the corresponding periodic window exists near the parameter. Contrary, if the minimum angle of an unstable periodic orbit at a parameter is large, the corresponding periodic window exists far from the parameter. Table 1 shows a relationship between minimum angles of unstable periodic orbits at $r = 60$ and parameter values of the end of the corresponding periodic windows. We select seven unstable periodic orbits with small minimum angles and six ones with relatively large minimum angles. If there exists an unstable periodic orbit which has a small (large) angle at $r = 60$, there is the corresponding window near the parameter $r = 60$. This fact gives us the idea that unstable periodic orbits are useful to detect parameters where the corresponding periodic windows exist.

minimum angle (degree)	period	r value at periodic window
0.9950	5.2497	60.4246164
1.2405	4.7767	60.25
1.4678	5.2558	60.4246165
1.5729	5.7184	60.51
2.1250	4.3212	61.31
2.1157	6.2262	61.34
2.3200	4.7986	61.63
4.6246	3.8578	63.35
7.9627	3.3820	86.40
15.1076	3.4090	118.13
15.3580	2.9155	126.52
17.4516	1.9409	154.43
31.0077	0.9773	312.96

Table 1: Relation between minimum angles of stable and unstable manifold at points on unstable periodic orbits at $r = 60$ and the position of the corresponding periodic window. Periodic windows corresponding to unstable periodic orbits which have small (large) angles exist near (away from) the parameter.

3 Summary

In this paper, we characterize local manifold structures of Lorenz attrctor by local manifolds of unstable periodic orbits and obtain two results on the relation between manifold structures and periodic windows. First, the minimum angle between stable and unstable manifolds at points on each unstable periodic orbit decreases monotonically as a parameter approaches to the corresponding periodic window and the angle becomes very small positive value at the ending point of the periodic window. This result suggests that the ending point of each periodic window is non-hyperbolic without any tangency structures. Second, when an unstable periodic orbit at a parameter has a low angle (high angle), the corresponding periodic window exists near (away from) the parameter. We note for the second result that periodic windows are not constructed completely in ascending order of an angle. This is because there are two unstable periodic orbits which have the same symbol sequences but are different ones. One is saddle and the other is node at the end of the periodic window. On the other hand, in the Lorenz system with classical parameters there are no unstable periodic orbits with the same symbolic sequences and thus periodic windows derive from the UPOs are made completely in ascending order of an angle. By using this result we can give a new approach to the first tangency problem of the Lorenz system which is one of the important problems in dynamical system theory [17]. We have also checked that a tangency structure is related to a period doubling bifurcation of node orbits of a saddle-node bifurcation, which is reported by another paper [17].

References

[1] E. Ott, *Chaos in Dynamical Systems* (Cambridge Univ. Press, Cambridge, 1993).

[2] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, (Westview Press, 2001).

[3] H.-W. Lorenz, *Nonlinear Dynamical Economics and Chaotic Motion* (Springer, 1989).

[4] J. Palis and F. Takens, *Hyperbolicity & sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge studies in advanced mathematics, **35** (Cambridge University Press, Cambridge, 1993).

[5] C. Bonatti, L. J. Diaz and M. Viana, *Dynamics Beyond Uniform Hyperbolicity, Encyclopedia of Mathematical Sciences* **102** (Springer-Verlad, Berlin, 2000).

[6] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc., **73** (1967), 747.

[7] T. Kai and K. Tomita, Statistical Mechanics of Deterministic Chaos - The case of One-Dimensional Discrete Process -, Prog. Theor. Phys., **64** (1980),1532.

[8] P. Cvitanović, Invariant Measurement of Strange Sets in Terms of Cycles, Phys. Rev. Lett., **61** (1988), 2729.

[9] C. Grebogi, E. Ott and J.A. Yorke, Unstable Periodic Orbits and the Dimensions of Multifractal Chaotic Attractors, Phys. Rev. A, **37** (1988), 1711

[10] T. Morita, H. Hata, H. Mori, T. Horita and K. Tomita, Spatial and Temporal Scaling Properties of Strange Attractors and Their Representations by Unstable Periodic Orbits, Prog. Theor. Phys. **79**, (1988), 296.

[11] Y-L. Lai, Y. Nagai and C. Grebogi, Characterization of the Natural Measure by Unstable Periodic Orbits in Chaotic Attractors, Phys. Rev. Lett., **79** (1997), 649.

[12] S. M. Zoldi, Unstable Periodic Orbit Analysis of Histograms of Chaotic Time Series, Phys. Rev. Lett., **81** (1998), 3375.

[13] M. J. Davis, R. S. MacKay and A. Sannami, Markov shifts in the Hénon family, Physica D, **52** (1991), 171.

[14] W. Tucker, The Lorenz attractor exists C. R. Acad. Sci. Paris Sèr. I Math., **328** (1999), 1197.

[15] W. Tucker, A rigorous ODE solver and Smale's 14th problem, Found. Comput. Math., **2** (2002), 53.

[16] Y. Saiki and M. U. Kobayashi, Numerical Identification of Nonhyperbolicity of the Lorenz Systems through Lyapunov Vectors, accepted to J. SIAM Lett.

[17] Y. Saiki and M. U. Kobayashi, Period doubling bifurcation as the origin of tangency in the Lorenz system, in preparation.

[18] V. Franceschini, C. Giberti and Z. Zheng, Characterization of the Lorenz attractor by unstable periodic orbits Nonlinearity, **6** (1993), 251.

[19] F. Ginelli, P. Poggi, A. Turchi, H. Chate, R. Livi and P. Politi, Characterizing dynamics with covariant Lyapunov vectors, Phys. Rev. Lett., **99**(13) (2007), 130601.

[20] V. I. Oseledec, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, Trans. Moskow Math. Soc., **19** (1968), 197.

[21] D. Ruelle, Ergodic theory of differentiable dynamical systems, Publ. Math. IHES, **50** (1979), 275.

[22] C. Sparrow, *The Lorenz equations: Bifurcations, chaos, and strange attractors*, (Springer-Verlag, New York, 1982).

[23] M. U. Kobayashi and Y. Saiki, Periodic window as the origin of non hyperbolicity in some chaotic systems, in preparation.